Monday, February 08, 2016
8:15 AM

Last time we tried to build an invariant $\tilde{\psi}\psi$ from spinors trying $\tilde{\psi} = \psi^*$ like we did with SU(2),

but... $\tilde{\psi}^T\psi = (\psi^*)^T\psi = \psi^\dagger\psi \to (\psi')^\dagger\psi' = \left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi\right)^\dagger e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$

$$= \psi^\dagger \underbrace{\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^\dagger} e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$$

Recall: $\sigma^{0i} = \frac{i}{2}\begin{pmatrix}\sigma_i & 0\\ 0 & -\sigma_i\end{pmatrix}$ $\sigma^{ij} = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix}\sigma_k & 0\\ 0 & \sigma_k\end{pmatrix}$

and $\sigma^{n\dagger} = \sigma^n$

But the $\sigma^{\mu\nu}$ are <u>not</u> all Hermitian!

In particular $\sigma^{0i\dagger} = -\sigma^{0i}$ Not-Hermitian
$\sigma^{ij\dagger} = \sigma^{ij}$ Hermitian

The reason $\psi^\dagger\psi$ worked for SU(2) is that the generators were all Hermitian.

But we can fix this with a different choice of dual: $\tilde{\psi} = i\gamma^0\psi^*$

Then: $\tilde{\psi}^T\psi = (i\gamma^0\psi^*)^T\psi$

$= i\psi^\dagger\gamma^{0T}\psi$

$= i\psi^\dagger\gamma^0\psi \to i(\psi')^\dagger\gamma^0\psi' = i\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi\right)^\dagger\gamma^0 e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$

$$= i\psi^\dagger\underbrace{\left(e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\right)^\dagger\gamma^0}_{\gamma^0 e^{-\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}} e^{\frac{i}{4}\sigma^{\mu\nu}\omega_{\mu\nu}}\psi$$

<u>You will show this in your HW</u>

Same! $= i\psi^\dagger\gamma^0\psi$

So in the end we define our dual spinor with $\tilde{\psi} = i\gamma^0\psi^*$ and the adjoint $\bar{\psi} \equiv \tilde{\psi}^T = i\psi^\dagger\gamma^0$

$\Downarrow$

$\bar{\psi}\psi$ is invariant

Or in other words $g = i\gamma^0 \Rightarrow \tilde{\psi} = g\psi$